15.053x, Optimization Methods in Business Analytics

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A glossary of notation and terms used in 15.053x

Weeks 1, 2, 3, 4, 5 and 6.

(The most recent week's terms are in blue).

NOTATION AND TERMINOLOGY

The purpose of this document is to provide a glossary of notation and terminology relevant to 15.053x. We will add notation and terminology throughout the semester, and we will update this document once a week. If you would like terms or notation added, contact the TA, Khizar Qureshi.

For a comprehensive (and mathematically advanced) glossary of mathematical terms used in optimization, see the Math Programming Glossary, which was developed by Harvey Greenberg.

MATHEMATICAL NOTATION

- $\sum_{i \in S} x_i$ = The sum of x_i where the sum is over all indices *i* in the set *S*. We refer to this type of notation as *summation notation*.
- $|x|$ = the absolute value of *x*. (This assumes that *x* is a single variable.)
- $\lfloor x \rfloor$ = the *floor* of *x*. That is, *x* rounded down to the nearest integer. For example, $\lfloor 2.3 \rfloor = 2$; $\lfloor -1.1 \rfloor = -2$; $\lfloor x \rfloor = x$ if *x* is an integer.
- $\lceil x \rceil$ = the *ceiling* of *x*. That is, *x* rounded up to the nearest integer. For example, $\lceil 2.3 \rceil = 3$; $\lceil -1.1 \rceil = -1$; $\lceil x \rceil = x$ if *x* is an integer.
- $x^+ = \max \{0, x\}$. This is often referred to as the *positive part* of *x.*
- $x^- = \min \{0, x\}$. This is often referred to as the *negative part* of *x*.

- ":" The symbol ":" is often used to mean "such that." For example, consider $\{(x, y) : 1 \le x \le 2, x + y \ge 0\}$. This is interpreted as "The set of points (x, y) such that x and y satisfy the following conditions: $1 \le x \le 2$ and $x + y \ge 0$}. Usually the conditions that need to be satisfied are separated by commas, but occasionally they would be separated by semicolons (";").
- (2, 4]: this is the set of real numbers x such that $2 < x \le 4$. When describing intervals over the reals, the parenthesis is used for an "open interval". That is, if a parenthesis is used, the interval does not contain the endpoint. A bracket -- "[" or "]" -- is used to indicate a closed interval. That is, the endpoint is included.
- $B(S, x', \varepsilon)$: The ϵ-ball in *S* centered at *x*'. See "ϵ-ball" below.

TYPES OF OPTIMIZATION MODELS.

By an *optimization model* (or *optimization problem*) we mean a problem in which there is a single objective function (max or min) subject to constraints. An alternative term that is commonly used is *mathematical program*. We also refer to them as *maximization problems* or *minimization problems*.

- *Linear Program*: an optimization model in which the objective is linear and the constraints are linear.
- *Mixed Integer Linear Program*: an optimization model in which the objective is linear and the constraints are linear, and some (or all) of the variables are constrained to be integer valued. It is called a *Pure Integer Program* if every variable is required to take on an integer value. It is called a *Binary Integer Program* (or a 0-1 Integer Program) if every variable is required to be 0 or 1.
- *Nonlinear Program*. This is the common name that refers to any possible optimization model. Remember that nonlinear programs include linear programs as a special case.

OTHER TERMINOLOGY

- 100% rule: This is a rule used in LP sensitivity analysis when more than one cost coefficient changes or when more than one RHS coefficient changes. If K coefficients change, one can determine allowable increases and decreases by dividing all of the allowable increases and decreases of the report by K. (There is a more general version of this rule in *Applied Mathematical Programming*.)

- Allowable decreases or increases. In the LP sensitivity report for cost coefficients, the allowable increase (resp., decrease) is the maximum increase (resp., decrease) in a single cost coefficient, with all other data being unchanged, such that there is no change in the optimal solution. In the LP sensitivity report for RHS coefficients, the allowable increase (resp., decrease) is the maximum increase (resp., decrease) in a single RHS coefficient, with all other data being unchanged, such that there is no change in the shadow prices.

- *Big M method*. In integer programming, this is a method that is used for modeling (i) logical constraints such as constraints involving "OR" or "IF-THEN." The big M method is also used for modeling fixed charges in the cost function. In practice, one needs to use a numerical value of M. In those cases, it helps to select a <u>minimal</u> value, that is, the smallest value of M which is guaranteed to work.

  Within linear programming the same term is used for a very different approach that helps to solve an LP when no initial feasible solution is known.

- *Bounded feasible region*. We say that a feasible region is *bounded* if there is some positive number M such that every decision variable is guaranteed to be between $-M$ and M. If a feasible region is not bounded, we say that it is *unbounded*.

- *CBC.* The solution algorithm that is freely available and is commonly used in conjunction with OpenSolver to solve linear programs and integer programs.

- *Concave function*. Suppose f is a function in which the domain D is a convex set. Then f is *concave* if the function $-f(x)$ is convex. That is, for every two points $(x, f(x))$ and $(y, f(y))$ on the "curve", the line segment joining these two points lies on or below the curve. Equivalently, for every two points $x, y \in D$, $f((1 - \lambda)\, x + \lambda\, y) \geq (1 - \lambda)\, f(x) + \lambda\, f(y)$.

- *Concave maximization problem*. This is an optimization problem in which the objective is to maximize a concave function, and the feasible region is a convex set. A local maximum for a concave maximization problem is guaranteed to be a global maximum.

- *Constraints*: Inequalities (or equalities) to impose limitations on the decision variables.

- *Convex function*. Suppose f is a function in which the domain D is a convex set. Then f is *convex* if for every two points $(x, f(x))$ and $(y, f(y))$ on the "curve", the line segment joining these two points lies on or above the curve. Equivalently, for every two points x, $y \in D$, $f((1 - \lambda)\, x + \lambda\, y) \leq (1 - \lambda)\, f(x) + \lambda\, f(y)$. The function f is called <u>strictly convex</u> if for every two distinct points $x, y \in D$, $f((1 - \lambda)\, x + \lambda\, y) < (1 - \lambda)\, f(x) + \lambda\, f(y)$.

- *Convex minimization problem*. This is an optimization problem in which the objective is to minimize a convex function, and the feasible region is a convex set. A local minimum for a convex minimization problem is guaranteed to be a global minimum.

- *Convex set*. A set *S* is *convex* if for every two points $p_1, p_2 \in S$, the line segment joining p_1 to p_2 is also in *S*. Equivalently, for all $\lambda \in [0,1]$, the point $(1 - \lambda)\, p_1 + \lambda\, p_2$ is in *S*.
  Note: the feasible region of a linear program is always convex.

- CPLEX. A great commercial solver for linear programs and integer programs. It was originally created by Bob Bixby (See Gurobi), and is now owned by IBM. It is free for students at accredited universities.

- *Decision variables*. The variables that represent the decisions or choices to be made. If you are using spreadsheet optimization, these variables are the values in *Changing Cells* or *Changing Variable Cells*.

- *Edge of the feasible region.* A line segment on the boundary of the feasible region that joins two extreme points. These extreme points are *adjacent*. (Every two extreme points can be joined by a line segment. For the two extreme points to be adjacent, the line segment must be on the boundary of the feasible region.)

- *ϵ-ball*: Suppose that *S* is a convex set and that $x' \in S$. Then the ϵ-ball in *S* centered at *x*', denoted as $B(S, x', \varepsilon)$, is the set of all points in *S* that are within a distance of ε from *x*'.

- *Euclidean norm. Suppose that* $y = y_1, y_2, \ldots, y_k$ is a vector. Then the *Euclidean norm* of *y* is

$$\|y\|_2 = \left(\sum_{i=1}^{k} y_i^2 \right)^{1/2} .$$

- *Excel Solver*. The optimization software that is included with Microsoft Excel. (With Google Sheets, the free software is called *Solver*.) It can be used to solve linear programs (simplex method) or integer programs (simplex method) or nonlinear programs (GRG Nonlinear).

- *Extreme point* (also called *corner points*). In two dimensional LPs, these are feasible points where two different constraints hold with equality. If we are solving a linear program with non-negativity constraints, and if there is some optimal solution, then there is an extreme point that is optimal. More general definition: A feasible point *x* of an LP is an extreme point if *x* is not the midpoint of two other feasible points.
  Two extreme points are adjacent if they are joined by an "edge," which is a line segment on the boundary of the feasible region.

- Extreme ray. It is a ray whose endpoint is an extreme point, and such that the ray lies on the boundary of the (infinite) feasible region.

- *Facility location problem*. A combinatorial problem in which there is a set of customers and also a set of locations where facilities may be located. A solution is feasible if it assigns facilities to locations (subject to some additional constraints) and if it assigns customers to facilities (possibly subject to some additional constraints). The objective is to minimize the combined costs of locating the facilities and assigning the customers.

- *Feasible*. A point is said to be *feasible* if it satisfies all of the constraints of the optimization model. (A point represents the assignment of values to each of the decision variables.) The *feasible region* is the set of all feasible points.
- *Fixed charge*. This refers to a cost function $f(\ )$ of the following form: $f(0) = 0$. $f(x) = d + cx$ if $x > 0$. We refer to d as the fixed charge. This type of cost function arises when there is a cost of initiating an activity, and a linear cost thereafter.
- *Free*. A decision variable x is called *free* if it can be either positive or negative. If a variable is free, we also say that it is *unconstrained in sign*.
- *Forcing constraint.* A forcing constraint is a constraint of an integer program that forces a binary variable to be 1 when some specified condition is satisfied. For example, if $0 \le x \le 100$, and if y is binary, then the constraint $x \le 100\ y$ (equivalently, $y \ge x/100$) is a forcing constraint. The variable y is forced to be 1 whenever $x > 0$.
- *Geometric method.* This refers to a method for solving a linear program in two dimensions. An isoprofit line is drawn on the graph. Then the line is moved parallel to itself in a way to improve the objective function. It is moved as far as possible while still having at least one feasible point.
- *Global minimum*. Suppose that P is the problem $\min\{f(x) : x \in S\}$. We say that x' is a *global minimum* for the problem P if (i) $x' \in S$, and (ii) $f(x') \le f(y)$ for all $y \in S$.
- *Graph coloring problem*. This is a combinatorial problem based on coloring the vertices of a graph G = (V, E). A coloring of the vertices of V is feasible if adjacent vertices receive distinct colors. The objective is to find a feasible coloring that uses the fewest number of colors. *Map coloring* is a special case in which one wants to assign colors to the regions of a map so that regions that share a common border have different colors.
- *Infeasible*. A point is said to be *infeasible* if it violates one or more constraints of the optimization model. An optimization model is said to be *infeasible* if there are no feasible points (equivalently, there are no solutions).
- *Integrality constraint*. A constraint stipulating that one or more variables of a model are required to be integer valued.
- Knapsack Problem. An integer program (usually binary) with a single linear constraint. The problem has been used to model the problem of putting items in a knapsack subject to a weight constraint, or selecting projects subject to a budget constraint, or selecting prizes at a game show (15.053 application).

- *Level set* of a function *f*. If *f* is a function, then the *level set* of $f(x)$ at α is: $\{x : f(x) \le \alpha\}$. If *f* is a convex function, then the level set of $f(x)$ at α is a convex set.
- *Local minimum.* Suppose that P is the problem $\min\{ f(x) : x \in S\}$. We say that *x'* is a *local minimum* for the problem P if (i) $x' \in S$, and (ii) $f(x') \le f(y)$ for all $y \in B(S, x', \epsilon)$, where $B(S, x', \epsilon)$ is the ϵ-ball in *S* centered at *x*'.
- *Non-negativity constraints*. The constraints that constrain variables to be greater than or equal to 0.
- *Objective Function*. In an optimization model, the goal is to either minimize or maximize the objective function.
- *One hundred percent rule*: See 100% rule (at beginning of Glossary)
- *OpenSolver.* Spreadsheet modeling software that can be used to set up an optimization problem and call an algorithm to solve it. OpenSolver is freely available on the web at www.OpenSolver.org. OpenSolver can, in principle, be used to model and solve optimization problems with any number of variables. (Excel Solver is limited to 200 variables.) In reality, extremely large problems may take up more memory than is available in your computer, and they may require too much time to solve. In 15.053x, we typically use CBC to solve linear and integer programs. In addition, OpenSolver works with other optimization software such as CPLEX and Gurobi.
- *Optimal solution*. A *solution* refers to a feasible point. Suppose that one is trying to solve a maximization problem, and that the objective function is *f*( ). A solution x^* is called *optimal* (or *maximal*) if for any other feasible solution *x*', $f(x^*) \ge f(x')$. If it were a minimization problem, then x^* would be called an *optimal* (or *minimal*) *solution* if for any other solution *x*', $f(x^*) \le f(x')$.
- *Parametric analysis*: measures the affect in the optimum solution and/or the optimum objective value as one parameter in the problem changes.
- *Piecewise linear function.* We will describe a function *f*( ) of one variable defined on a domain $D = [0, U]$. A (continuous) piecewise linear function has the following property: there are points $a_1, \ldots, a_K$ for some K such that (i) $a_1 = 0$, (ii) $a_K = U$, and (iii) the function *f*( ) is linear on the closed interval $[a_i, a_{i+1}]$ for each $i = 1$ to K-1. It is also possible to define non-continuous piecewise linear functions in which the linear pieces do not necessarily share their endpoints.
- *Portfolio optimization*. A class of problems in Finance. A manager needs to create a portfolio of investments so as to maximize the return of the portfolio while minimizing the variance of the portfolio. A common variant of this problem is to minimize the variance while ensuring that the return of the portfolio meets or exceeds a threshold

value. This variant of the problem is a quadratic convex minimization problem.

- *Pricing out* a variable x_j. This is a method for computing the reduced cost of x_j. One multiplies the inner produce of the column coefficients of x_j by the shadow prices, and subtracts this inner product from the profit of x_j.
- *Reduced cost* of a variable x_j. This is the shadow price of the constraint $x_j \geq 0$.
- *Redundant constraint*. A constraint of an optimization problem with the following property: if the constraint is deleted, then the feasible region does not change. It is necessary to understand redundant constraints as part of the big M method for modeling integer programs.
- RHS: abbreviation for Right Hand Side.
- *Second-Order Cone Program* (SOCP): The SOCP is an optimization problem in which the objective function is linear, and constraints are either linear or they are of the form: $\| y \|_2 + a^t x \leq b$ where $\| y \|_2$ is the Euclidean norm of *y*, and $a^t x$ is a linear function of *x*, and *b* is a scalar.
- Sensitivity analysis: measures how sensitive the optimum solution and optimum objective value is to changes in the data. For linear programs, there is a sensitivity analysis report that provides information on changes in the optimum solution when a single cost coefficient changes. There is another report that provides information on how the optimal objective value changes when a single right hand side coefficient changes.
- *Set covering problem.* A combinatorial optimization problem in which there are n subsets of a set S. A collection of subsets is feasible if the union of the subsets is S. The objective is to find a minimum size feasible subset (or possibly a minimum cost feasible subset.)
- *Set packing problem.* A combinatorial optimization problem in which there are n items to be "packed". There is also a list of forbidden pairs of items. It is not permitted to pack both items in any of the pairs. The objective is to find a maximum size feasible subset (or possibly a maximum value feasible subset.)
- Shadow price. The shadow price of a constraint of a linear program is the increase in the optimal objective value per unit increase in the RHS coefficient.
- *Simplex Algorithm*. The most commonly used method for solving linear programs. It was developed by George Dantzig in 1947. It finds an optimal solution iteratively. It starts at an extreme point solution. It then moves to an adjacent extreme point solution whose objective value is better. If there is no adjacent extreme point that is better, then (1) there is an "extreme ray" along which the objective value improves infinitely (and thus the optimal solution value is infinite) or else (2) the current extreme point is optimal.

- Solution. Typically a *solution* refers to a feasible point of an optimization model. The term "infeasible solution" sounds like a paradox. But, the term *infeasible solution is widely used to* refer to a point that is infeasible. That is, it is not a solution.
- *Unbounded*. We say that a feasible region is *unbounded* if it is not bounded. That is, for any positive number *M*, there is some feasible solution x' such that some variable of x' has absolute value larger than *M*. We say that the optimal objective value of a maximization problem is *unbounded from above* if there is a sequence of feasible solutions whose objective values goes off to (converges to) ∞. Similarly, we say that the optimal objective value of a minimization problem is unbounded from below if there is a sequence of feasible solutions whose objective values converge to -∞.