15.481x – FINANCIAL MARKET DYNAMICS AND HUMAN BEHAVIOR

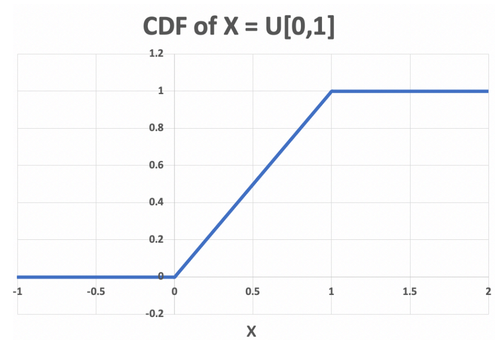
MIT MANAGEMENT SLOAN SCHOOL

Maximum of n IID RVs

REFRESHER: CDF, PDF AND EMPIRICAL CDF

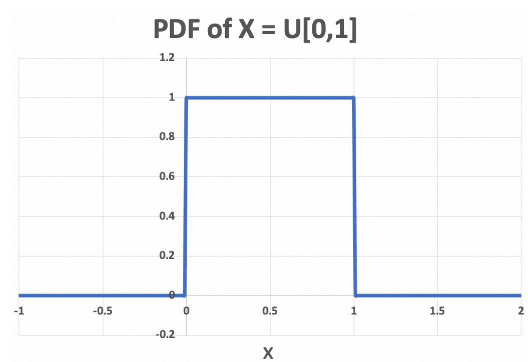
CDF: Cumulative Distribution Function

- Is defined for **<u>all</u>** random variables!
- For all $x \in \mathbb{R}$: $\mathrm{F}_X(x) = \Pr(\mathrm{X} < x)$



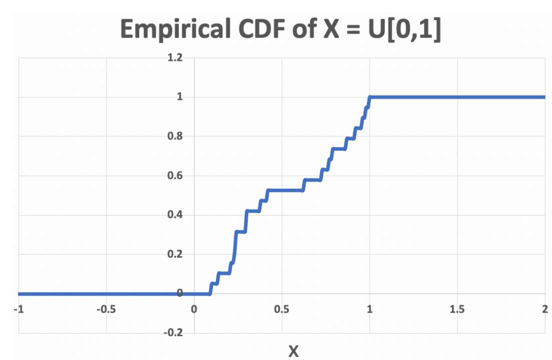
PDF: Probability Density Function

- Defined for **<u>continuous</u>** random variables!
- $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}_X(x)$ wherever F_X is differentiable.



Empirical CDF: used to estimate the CDF

- For all $x \in \mathbb{R}$: $\hat{\mathrm{F}}_X(x) = \frac{\#\text{ of X observed in } (-\infty,\ x]}{\text{Total } \#\text{ of X observed}}$



MAXIMUM OF IID RANDOM VARIABLES

Setting:

- **Random Variables:** Given n i.i.d. random variables $X_i \sim F_X$ (distribution)
- **Goal:** Find the distribution of the r.v. $M = \max(X_1, \dots, X_n)$

Example:

$$\text{Let } n = 7 \text{ and } X_i \sim \text{Uniform}[0,1]$$

- Run 3 simulations and record M each time:

i	X_i							M
	1	2	3	4	5	6	7	
Trial 1	0.456	0.064	0.357	0.091	0.733	0.944	0.188	0.944
Trial 2	0.404	0.776	0.737	0.414	0.204	0.238	0.342	0.776
Trial 3	0.114	0.549	0.068	0.722	0.508	0.764	0.722	0.764

MAXIMUM OF IID RANDOM VARIABLES

Setting:

- **Random Variables:** Given n i.i.d. random variables $X_i \sim F_X$ (distribution)
- **Goal:** Find the distribution of the r.v. $M = \max(X_1, \dots, X_n)$

Proof:

$$
\begin{aligned}
F_M(x) &= \Pr(M \le x) = \Pr(\max(X_1, \dots, X_n) \le x) \\
&= \Pr(X_1 \le x, \dots, X_n \le x) \\
&= \Pr(X_1 \le x) \times \dots \times \Pr(X_n \le x) \\
&= \Pr(X \le x)^n \\
&= F_X(x)^n
\end{aligned}
$$

Result:

$$F_M(x) = F_X(x)^n$$

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Minimum of n IID RVs

MINIMUM OF IID RANDOM VARIABLES

Setting:

- **Random Variables:** Given n i.i.d. random variables $X_i \sim F_X$ (distribution)
- **Goal:** Find the distribution of the r.v. $m = \min(X_1, \ldots, X_n)$

Example:

$$\text{Let } n = 7 \text{ and } X_i \sim \text{Uniform}[0,1]$$

- Run 3 simulations and record m each time:

i	X_i							m
	1	2	3	4	5	6	7	
Trial 1	0.811	0.306	0.209	0.881	0.382	0.967	0.651	0.209
Trial 2	0.998	0.812	0.671	0.11	0.517	0.502	0.426	0.11
Trial 3	0.356	0.784	0.485	0.773	0.452	0.475	0.195	0.195

MINIMUM OF IID RANDOM VARIABLES

Setting:

- **Random Variables:** Given n i.i.d. random variables $\mathrm{X}_i \sim \mathrm{F}_\mathrm{X}$ (distribution)
- **Goal:** Find the distribution of the r.v. $\mathrm{m} = \min(\mathrm{X}_1, \dots, \mathrm{X}_n)$

Proof:

$$
\begin{aligned}
\mathrm{F}_\mathrm{m}(x) = \Pr(\mathrm{m} \leq x) = 1 - \Pr(\mathrm{m} \geq x) &= 1 - \Pr(\min(\mathrm{X}_1, \dots, \mathrm{X}_n) \geq x) \\
&= 1 - \Pr(\mathrm{X}_1 \geq x, \dots, \mathrm{X}_n \geq x) \\
&= 1 - \Pr(\mathrm{X}_1 \geq x) \times \dots \times \Pr(\mathrm{X}_n \geq x) \\
&= 1 - [1 - \Pr(\mathrm{X} \leq x)]^n \\
&= 1 - [1 - \mathrm{F}_\mathrm{X}(x)]^n
\end{aligned}
$$

Result:

$$\mathrm{F}_\mathrm{m}(x) = 1 - [1 - \mathrm{F}_\mathrm{X}(x)]^n$$

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Order Statistics

JOINT DISTRIBUTION FOR ORDER STATISTICS

Setting:

- **Random Variables:** Given n i.i.d. random variables $\mathrm{X}_i \sim \mathrm{F}_\mathrm{X}$ (distribution)
- **Order Statistic:** Sort the variables $\mathrm{X}_1, \ldots, \mathrm{X}_n$ and label them $\mathrm{U}_1, \ldots, \mathrm{U}_n$
- **Goal:** Find the distribution of the r.v.s $\mathrm{U_i}$

Note: $\mathrm{U}_1 \leq \mathrm{U}_2 \leq \cdots \leq \mathrm{U}_n$

Theorem: we can get the joint PDF

$$f_{\mathrm{U}_1,\mathrm{U}_2,\ldots,\mathrm{U}_n}(u_1, u_2, \ldots, u_n) = \begin{cases} n! \cdot \prod_{i=1}^{n} f_\mathrm{X}(u_i), & \text{for } u_1 \leq u_2 \leq \cdots \leq u_n. \\ 0, & \text{otherwise.} \end{cases}$$