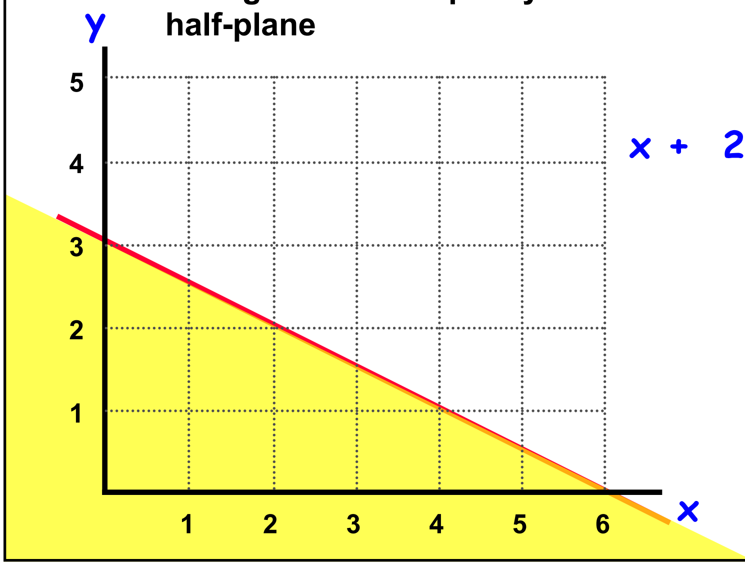
Graphing the feasible region of a linear program

A Two Variable Linear Program (a variant of the DTC example)

max	$z = 3x + 5y$		
s.t	$2x + 3y \leq 10$	(1)	Constraints
	$x + 2y \leq 6$	(2)	
	$x + y \leq 5$	(3)	
	$x \leq 4$	(4)	
	$y \leq 3$	(5)	
	$x, y \geq 0$	(6)	

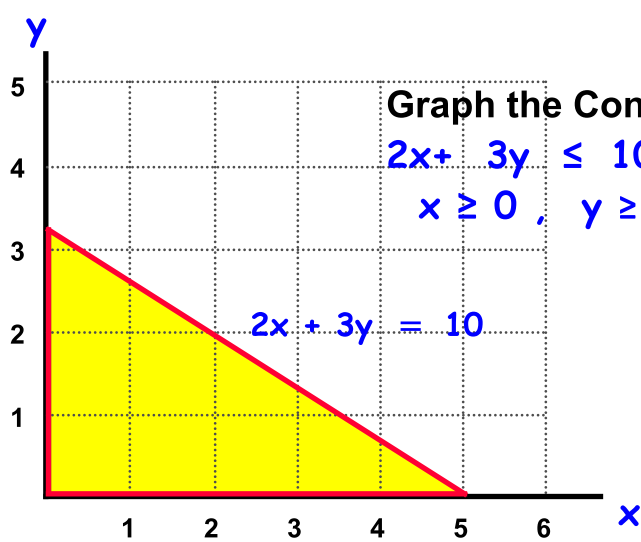
Inequalities

A single linear inequality determines a unique half-plane



$x + 2y \leq 6$

Graphing the feasible region



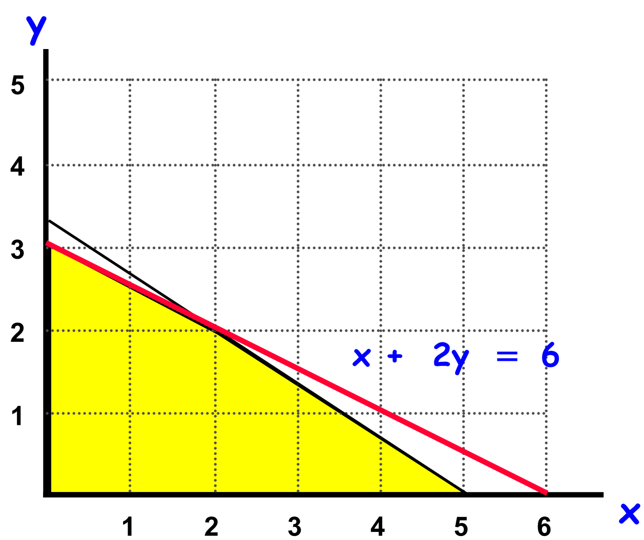
Graph the Constraints:

$2x + 3y \leq 10$ **(1)**

$x \geq 0$, $y \geq 0$. **(6)**

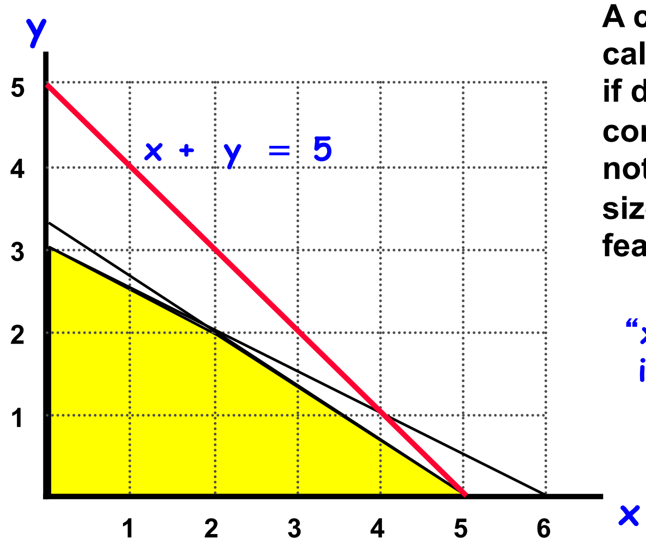
Add the Constraint:

$x + 2y \leq 6$ (2)



Add the Constraint:

$x + y \leq 5$



A constraint is called ***redundant*** if deleting the constraint does not increase the size of the feasible region.

"$x + y \leq 5$" is redundant

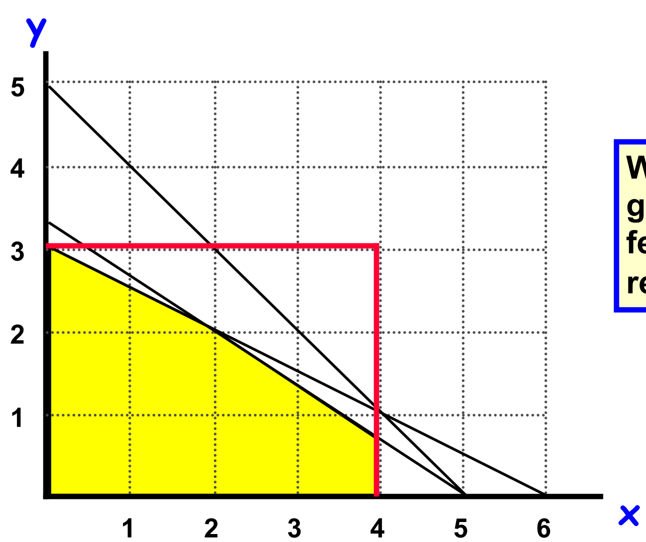
On redundant constraints

- A redundant constraint can be removed
  - usually not a good idea
    - It can be confusing to others
    - The constraint may be needed when doing parametric analysis
    - LP solvers are incredibly fast

- Redundancy: very useful concept in modeling for <u>integer programming</u>.

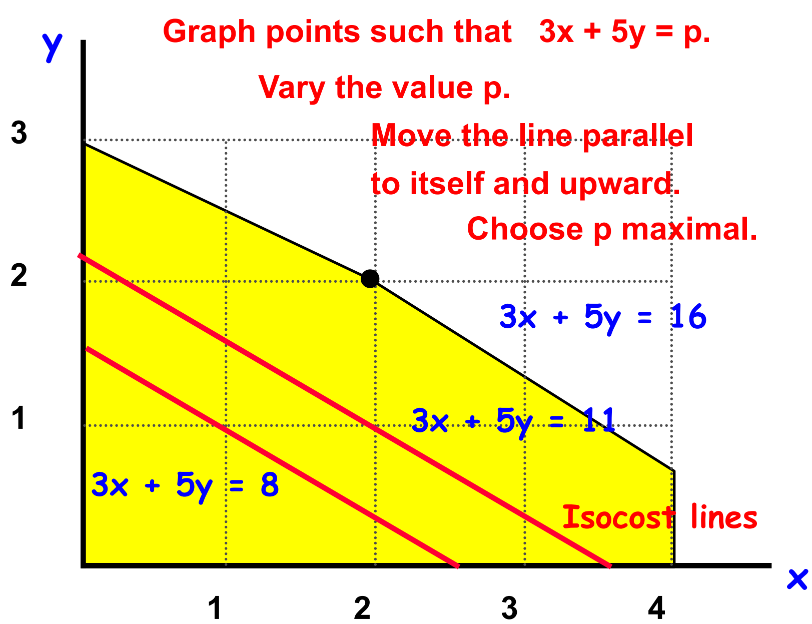
Add the Constraints:

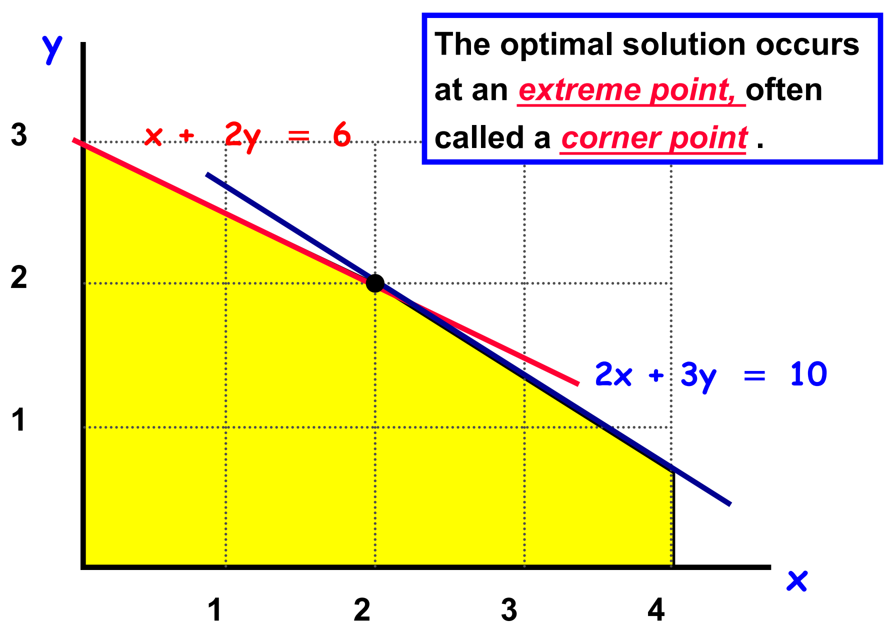
$x \leq 4; \quad y \leq 3$



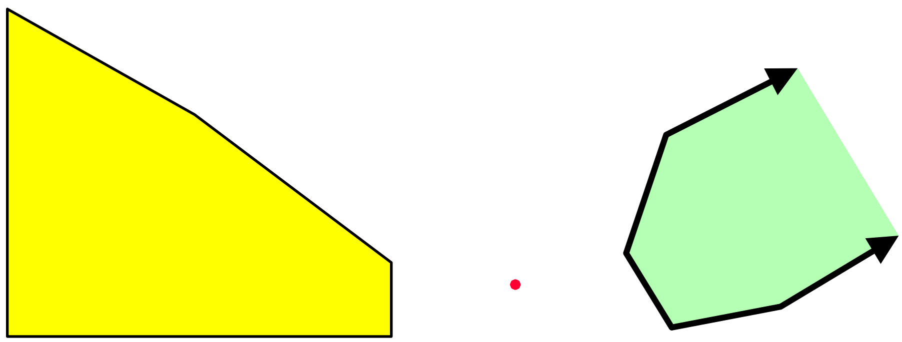
We have now graphed the feasible region.

The geometrical method for optimizing $3x + 5y$

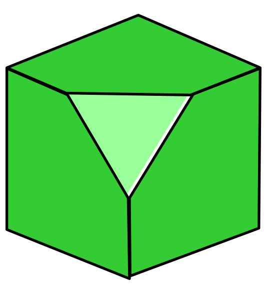


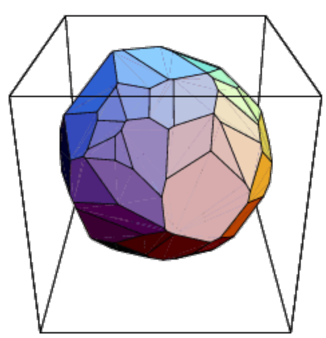


Some 2 dimensional linear programs



Some 3-dimensional linear programs





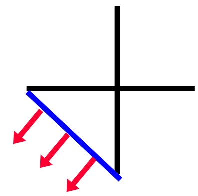
mathworld.wolfram.com/ ConvexPolyhedron.html

Is there always an optimal solution?

Different types of LPs

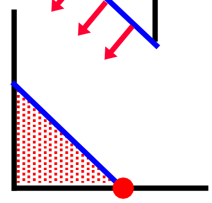
Infeasible LP's: that is, there is no feasible solution.

max x
s.t. $x + y \le -1$
$x \ge 0, y \ge 0$



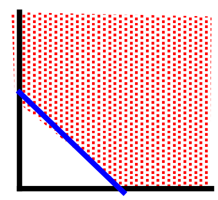
LPs that have an optimal solution.

max x
s.t. $x + y \le 1$
$x \ge 0, y \ge 0$



LPs with unbounded objective. (For a max problem this means unbounded from above.)

max x
s.t. $x + y \ge 1$
$x \ge 0, y \ge 0$



Any other types

Theorem. ***If the feasible region is non-empty and bounded, then there is an optimal solution.***

This is true when all of the inequalities are "≤ constraints", as opposed to "< constraints".

e.g., the following problem has no optimum

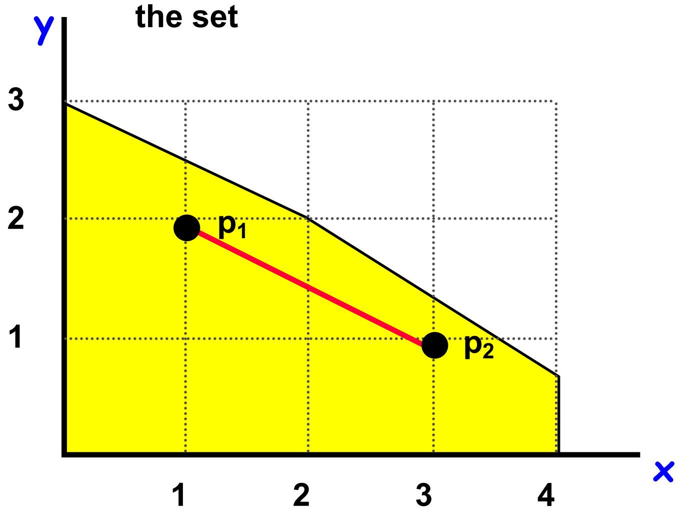
Maximize x

subject to $0 < x < 1$

Convex sets

Convex Sets

A set S is *convex* if for every two points in the set, the line segment joining the points is also in the set

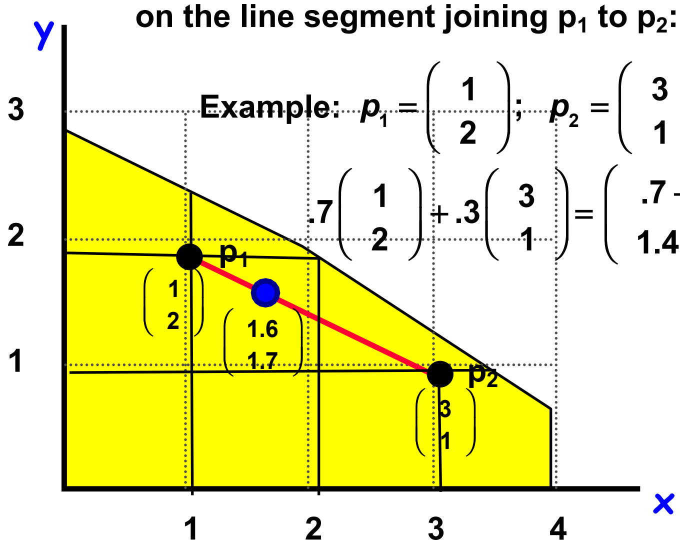


Convex Sets

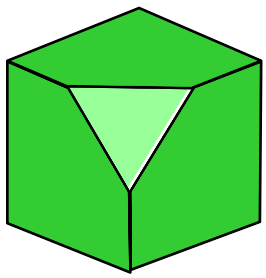
For every value $q \in [0, 1]$, the following is a point on the line segment joining p_1 to p_2: $(1 - q)p_1 + qp_2$.

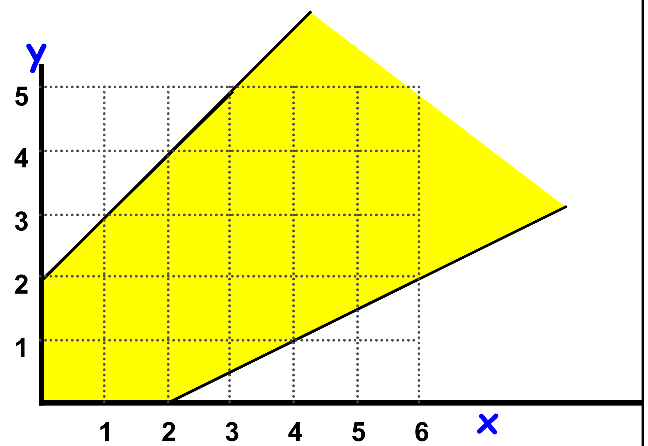
Example: $p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad p_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}; \quad q = .3$

$$.7\begin{pmatrix} 1 \\ 2 \end{pmatrix} + .3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} .7 + .9 \\ 1.4 + .3 \end{pmatrix} = \begin{pmatrix} 1.6 \\ 1.7 \end{pmatrix}$$



The feasible region of a linear program is convex.

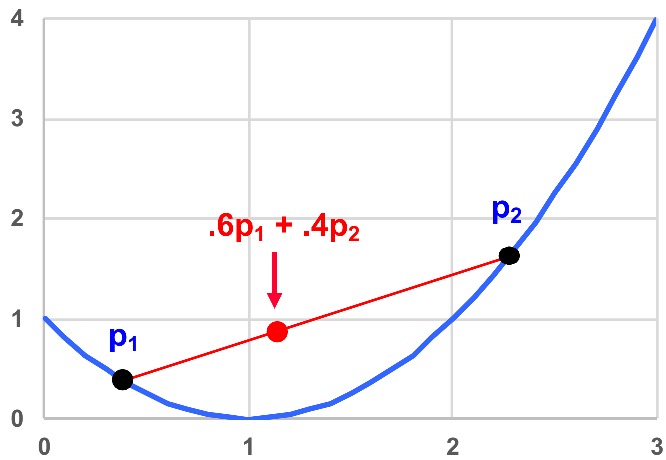




Convex functions

Geometry of convex functions

A function f is convex if for every points p_1 and p_2 on the curve, the line segment joining p_1 to p_2 lies on or above the curve.



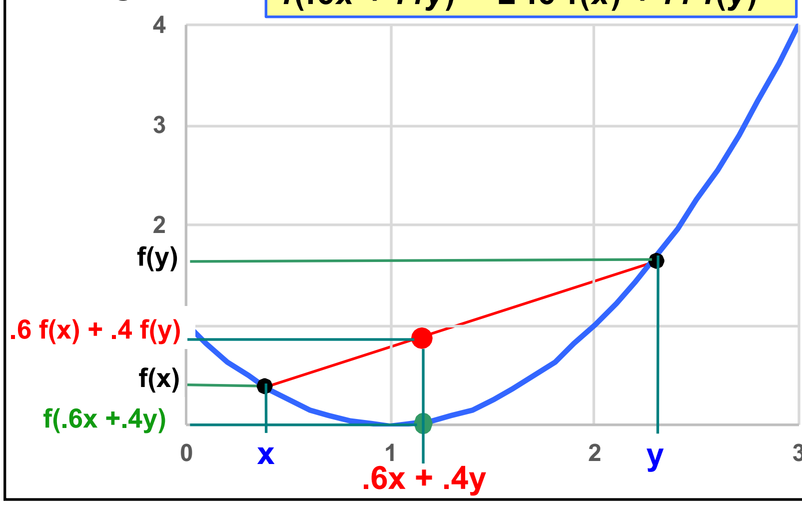
Suppose f is defined on convex domain D.

A function f is convex if for all x, y ∈ D, and all λ ∈ [0,1]

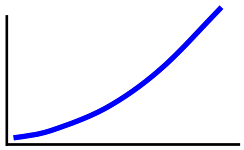
$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

e.g., λ = .4

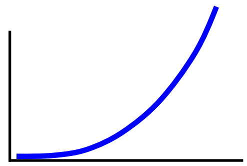
$$f(.6x + .4y) \leq .6\, f(x) + .4\, f(y)$$



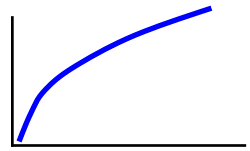
Which functions are convex?



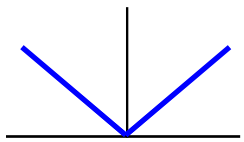
f(x) = x^2



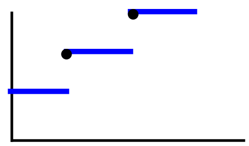
f(x) = x^3 for $x \geq 0$



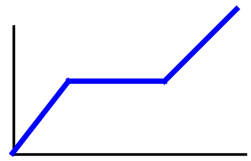
f(x) = $x^{.5}$



f(x) = |x|



Step Function



whatever

Yes No

Why convexity?

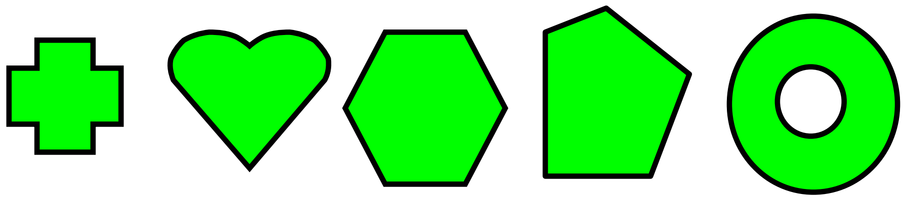
If one is minimizing, then a non-linear programming solver will (in general) find an optimal solution if

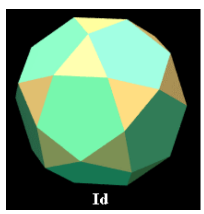
- The function being minimized is a convex function
- The feasible region is convex

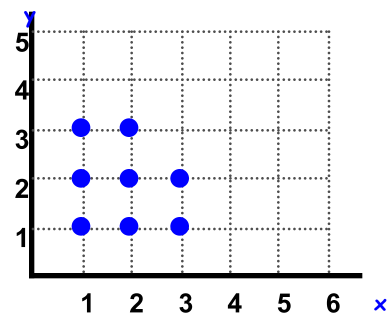
More on this is Week 6 of this course.

Convex or not?

Which of the following are convex ? or not ?







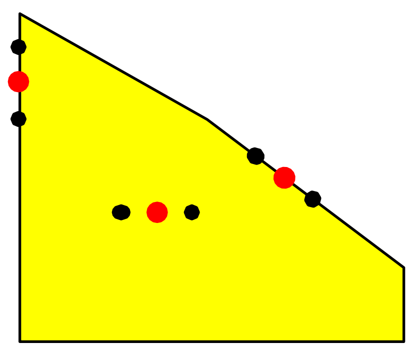
Why should we care about convex regions?

- It's much harder to find the optimum if the feasible region isn't convex.
  - We do well in solving integer programs, much better than one might expect
  - It can be incredibly hard to solve other nonlinear programs if the feasible region isn't convex.

Extreme Points
(Corner Points)

Extreme Points

- An ***extreme point*** (also called a ***corner point***) of the feasible region is a point that is not the midpoint of two other points of the feasible region. (They are only defined for convex sets.)



The red "points" are not extreme points.

Extreme Points

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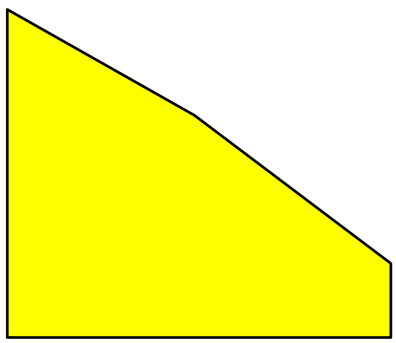

Where are the extreme points of this feasible region?

Extreme Points

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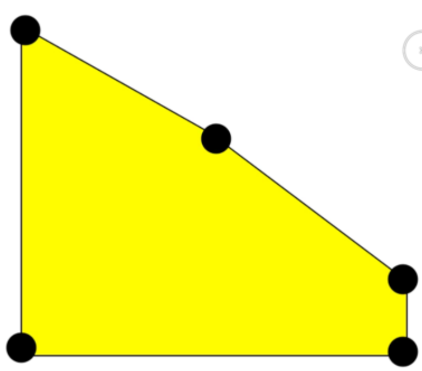
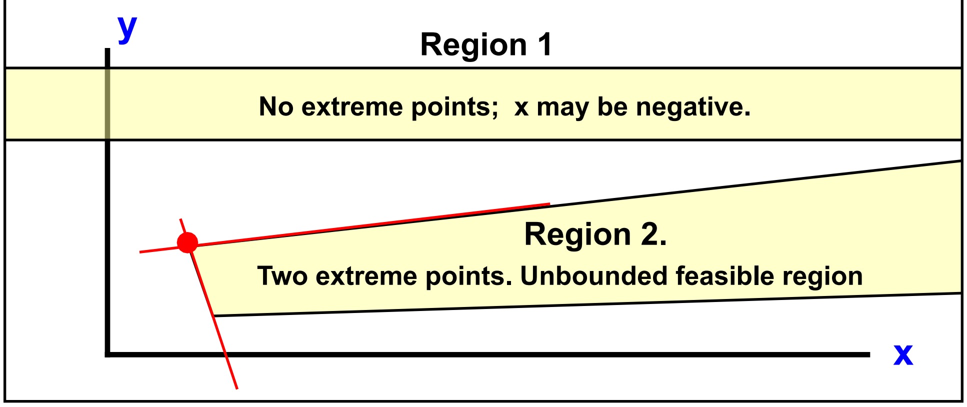


Where are the extreme points of this feasible region?

Facts about extreme points.

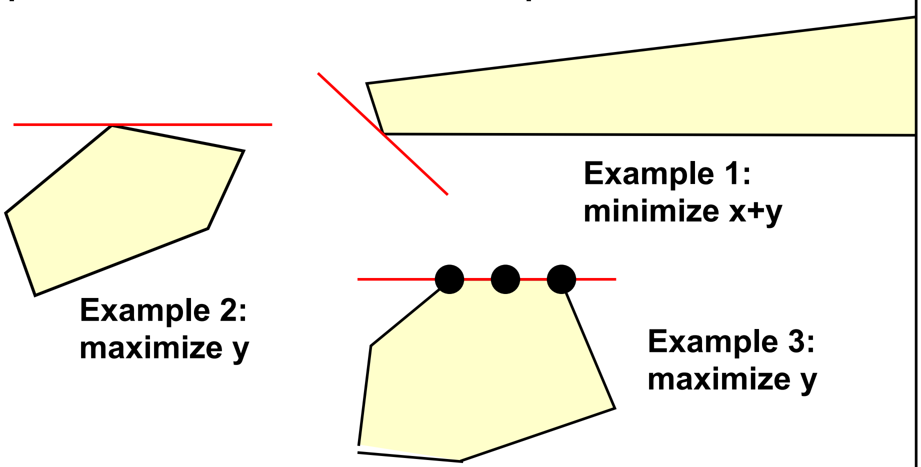
If every variable is non-negative, and if the feasible region is non-empty, then there is an extreme point.

In two dimensions, an extreme point is at the intersection of two equality constraints.



Optimality at extreme points for LPs

If a feasible region of a linear program has an extreme point, and if it has an optimal solution, then there is an optimal solution that is an extreme point.

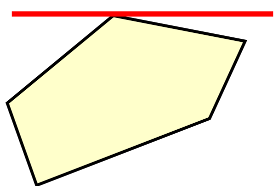


The importance of extreme points

The solution to an LP in 2 dimensions can always be found by solving 2 equations with 2 variables.

The solution to an LP in 3 dimensions can always be found by solving 3 equations with 3 variables.

The solution to an LP in 100 dimensions can always be found by solving 100 equations with 100 variables….



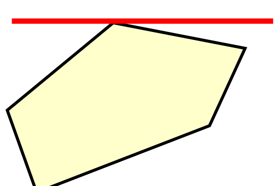
Example 2: maximize y

Extreme points tell us a lot about the "structure of the LP solution."

The importance of extreme points

If one perturbs the data by a very small amount, usually the optimum solution is found by solving the same (but perturbed) set of equations.

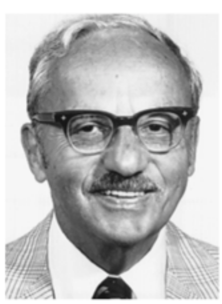
This observation is the starting point for LP sensitivity analysis.



Example 2: maximize y

The simplex method (next video) works by moving from extreme point to extreme point, always improving the objective function.

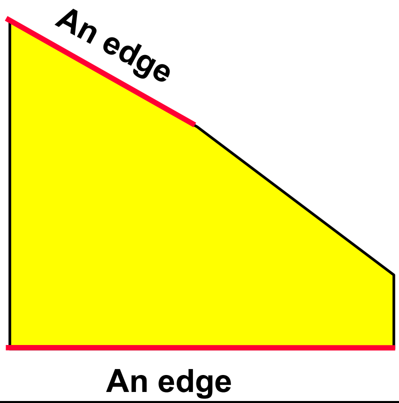
The simplex method



George Dantzig.
The creator of the simplex method for linear programming.

Edges of the feasible region

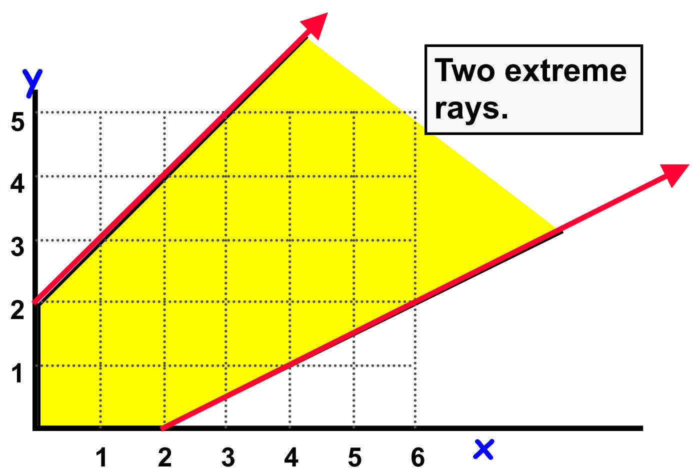
In two dimensions, an edge of the feasible region is one of the line segments making up the boundary of the feasible region. The endpoints of an edge are extreme points.



In two dimensions, it is a (bounded) equality constraint.

Extreme Rays

- An extreme ray is like an edge, but it starts at an extreme point and goes on infinitely.

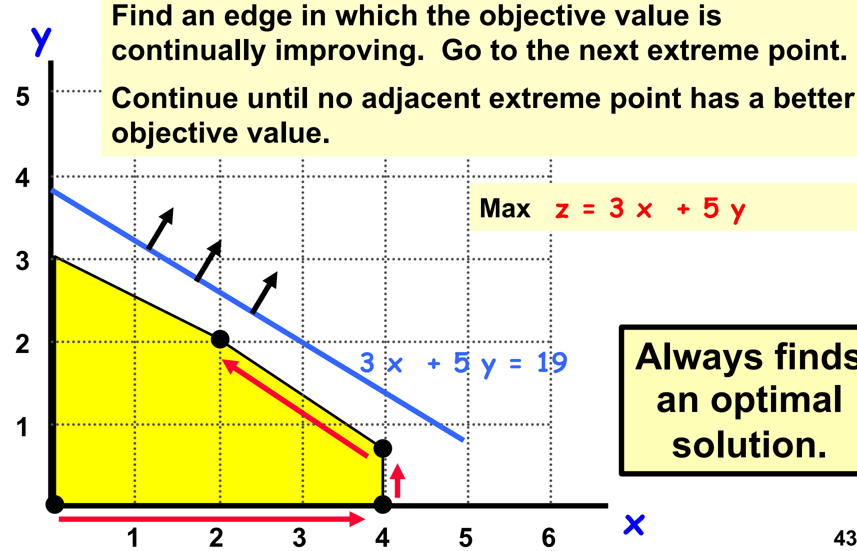


The Simplex Method
(assume bounded feasible region)

Start at any feasible extreme point.

Find an edge in which the objective value is continually improving. Go to the next extreme point.

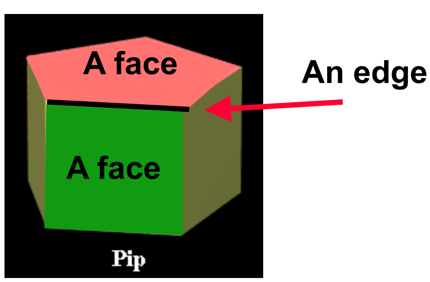
Continue until no adjacent extreme point has a better objective value.



Always finds an optimal solution.

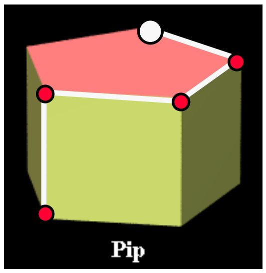
Edges of the feasible region

In three dimensions, an **edge** of the feasible region is one of the line segments making up the framework of a polyhedron. The edges are where the faces intersect each other. A **face** is a flat region of the feasible region.



In two dimensions it is a bounded intersection of two equality constraints.

The Simplex Method



Pentagonal prism

Note: in three dimensions, the "edges" are the intersections of two constraints. The corner points are the intersection of three constraints.

Simplex method

Developed in 1947 by George Dantzig

Still the method of choice today for solving linear programs.

Determines whether an LP is infeasible.
Finds an optimal solution if there is one.
Proves unbounded if an LP is unbounded.

VERY fast in practice.